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Generating functions for higher-order interaction terms in the IBA Hamiltonian

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Abstract. The generating function for the number of independent Hermitian SO(3) scalar operators in the enveloping algebra of U(6) restricted to totally symmetric U(6) representations is constructed. This predicts the number of interaction terms that may appear in the most general Hamiltonian for the interacting boson approximation (IBA) model. Then a complete analysis up to cubic interaction terms is given.

1. Introduction

In the original interacting boson model (IBM), initially introduced by Arima and Iachello (1976, 1978, 1979), a dynamical symmetry arises whenever the Hamiltonian H can be written in terms of invariants only of maximal subgroups $G \subset U(6)$. In these papers, the Hamiltonian was an expression up to second order in the U(6) generators. The reason for this is that only one- and two-body interactions between the s and d bosons were maintained. Recently, there has been some interest in introducing higher-order interactions between the bosons. In the SU(3) chain, three-body interactions were introduced in the Hamiltonian by Vanden Berghe *et al* (1985), and this gave rise to a much better approximation of the energy spectrum as well as to removal of the degeneracy which originally existed for members of the β and γ bands.

For one- and two-body interactions, it is well known (Iachello 1980) that the Hamiltonian can be written in terms of Casimir invariants of the subgroups U(5), SU(3), SO(6), SO(5) and SO(3). When analysing higher-order interaction terms, a number of questions arise. How many terms are contained in the most general higher-order Hamiltonian of a given degree for the interacting boson approximation (IBA) model? How many of those terms survive when a phenomenological analysis of its eigenvalue spectrum is performed? Can all remaining terms be expressed as dynamical group or subgroup invariants, or should one introduce so-called mixed-symmetry operators? These problems are thoroughly probed in the present paper.

In order to study the higher-order body interactions systematically, one has to answer the following group theoretical question: what is the structure of the centre of SO(3) in the enveloping algebra of U(6)? This problem can be tackled by means of the generating function (GF) technique. In fact, some partial results have been obtained already in a recent letter (Van der Jeugt 1986, hereafter referred to as I). In this letter,

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the three symmetry chains of U(6) were studied separately, and for each chain an integrity basis of possible higher-order terms was given. In fact, this implied the construction of the 'degenerate' GF for the number of elements in the enveloping algebra of G (where G is one of SU(3), U(5) or SO(6)) that commute with the SO(3) basis elements. The notion of a 'degenerate' GF was first introduced by Giroux *et al* (1984). It is actually a GF for certain objects acting on a class of degenerate representations of a Lie algebra only. A class of degenerate representations is characterised by the fact that a fixed set of Dynkin labels are always zero. A simple example is for instance the class of symmetric irreps of U(6), for which the Lie algebra representations of A_5 are labelled by $(N, 0, 0, 0, 0)$.

It is the aim of the present paper to establish a GF for the number of SO(3) scalars in the degenerate enveloping algebra (i.e. the enveloping algebra acting only on symmetric representations) of U(6). It is clear that the results obtained in I are also of use here, but do not yield a complete answer to the above problem.

The outline of the paper is as follows: in § 2 the Lie algebra of the IBA model is introduced, together with its subalgebras. In § 3 we construct the GF for the number of Hermitian SO(3) scalar operators in the degenerate enveloping algebra of U(6). In order to investigate which subalgebra invariants can be used in the IBA Hamiltonian, the (degenerate) enveloping algebras of SU(3), U(5) and SO(6) are discussed in § 4 and special attention is paid to the cubic interaction terms in § 5. Finally, an analysis is given of the most general IBA Hamiltonian up to third order in § 6.

2. The algebra of U(6)

It is well known that the generators of U(6) can be realised in terms of s and d boson creation and annihilation operators as follows:

$$\begin{array}{lll} [d^+ \times \tilde{d}]^{(j)} & (j = 0, 1, 2, 3, 4) & \\ {[s^+ \times \tilde{d}]^{(2)}} & [d^+ \times \tilde{s}]^{(2)} & [s^+ \times \tilde{s}]^{(0)}. \end{array} \tag{2.1}$$

Herein d^+_μ $(\mu = -2, -1, 0, 1, 2)$, $\tilde{d}_\mu = (-1)^\mu d_{-\mu}$ $(\mu = -2, -1, 0, 1, 2)$, s^+ and $\tilde{s} = s$ are SO(3) spherical tensor operator components satisfying

$$[d_\mu, d^+_\nu] = \delta_{\mu\nu} \qquad [s, s^+] = 1 \tag{2.2}$$

and all other commutators are equal to zero.

In this context the following operators are introduced (Iachello 1980):

$$\begin{array}{lll} T^{(j)} = [d^+ \times \tilde{d}]^{(j)} & (j = 1, 2, 3, 4) & \\ Q^{(2)} = [d^+ \times \tilde{s} + s^+ \times \tilde{d}]^{(2)} - \frac{1}{2}\sqrt{7}[d^+ \times \tilde{d}]^{(2)} & & \\ p^{(2)} = [d^+ \times \tilde{s} + s^+ \times \tilde{d}]^{(2)} & & \\ \hat{n}_d = \sqrt{5}[d^+ \times \tilde{d}]^{(0)} & \hat{n}_s = [s^+ \times \tilde{s}]^{(0)} & \hat{N} = \hat{n}_d + \hat{n}_s. \end{array} \tag{2.3}$$

Clearly, $\hat{N}$ is the U(6) number operator counting the total number $N = n_d + n_s$ of d and s bosons, whereas $\hat{n}_d$ (respectively $\hat{n}_s$) is the U(5) (respectively U(1)) number operator counting the number of d bosons n_d (respectively of s bosons n_s). Furthermore the operators $L_\mu = \sqrt{10}T^{(1)}_\mu$ $(\mu = -1, 0, 1)$ generate the physical angular momentum subalgebra SO(3). The three maximal dynamical symmetry subalgebras U(5), SU(3) and SO(6) are generated by the operator subsets $\{\hat{n}_d, T^{(1)}, T^{(2)}, T^{(3)}, T^{(4)}\}$, $\{T^{(1)}, Q^{(2)}\}$

and $\{T^{(1)}, T^{(3)}, P^{(2)}\}$ respectively. SO(5) is generated by $\{T^{(1)}, T^{(3)}\}$ and is obviously a subalgebra of U(5) and SO(6). Note that the generators of SU(5) (respectively SU(6)) are obtained from those of U(5) (respectively U(6)) by deleting $\hat{n}_d$ (respectively $\hat{N}$).

The standard IBA Hamiltonian is built from Hermitian SO(3) scalars in the U(6) enveloping algebra which are of first or second degree in the U(6) generators. In the present paper higher-order elements in the enveloping algebra of U(6) are investigated which still commute with the SO(3) basis elements. Moreover, if we think of these terms as being possible candidates for extending the Hamiltonian, we should also keep in mind that these elements must be Hermitian. This is in fact a further restriction of the problem, as we shall see in the following section.

3. Degenerate generating functions for U(6)

It is the aim of this section to construct a general formula for the number of independent Hermitian n-body interaction terms in the interacting boson model. There are two ways to proceed. On the one hand the techniques of I can be used by investigating the structure of the degenerate enveloping algebra of U(6). On the other hand the number of interaction terms is simply equal to the number of matrix elements between states of the same angular momentum l (because interaction terms must be SO(3) scalar operators). The two techniques give rise to similar calculations and the same results. Here, we shall follow only the second method, since it is much easier to understand.

In order to illustrate the second technique, we first give an example. The number of independent two-body interaction terms can be obtained by listing the available angular momenta coming from $(d+s)^2$, namely $l=0, 2, 4$ from d^2, $l=2$ from ds and $l=0$ from s^2. In total this gives $l=0^2, 2^2, 4$. Then the number of independent two-body terms (counting the four cells in the 2×2 matrix between $l=0$ states, etc) is $4+4+1=9$. Requiring Hermiticity implies the same counting but with Hermitian matrices. Hence the number of independent Hermitian two-body interaction terms is $3+3+1=7$. Similarly, the available angular momenta coming from $(d+s)^3$ are $l=0^3, 2^3, 3, 4^2, 6$. Thus, the number of independent three-body interactions is $9+9+1+4+1=24$, whereas the number of independent Hermitian three-body terms is $6+6+1+3+1=17$.

Now we intend to construct a general formula for the number of (Hermitian) n-body terms. From the above-mentioned examples it is clear that first one has to consider the angular momentum contents of the totally symmetric U(6) representation labelled by $[n]$ (i.e. the Lie algebra representation of A_5 with Cartan labels $(n, 0, 0, 0, 0)$). A generating function for the angular momentum states contained in $[n]$ is obtained from the branching rule GF for SO(5) $\rightarrow$ SO(3) (Gaskell *et al* 1978) of the form

$$G(V, L)=\frac{(1+V^3L^3)}{(1-V)(1-V^2)(1-V^3)(1-VL^2)(1-V^2L^2)}. \tag{3.1}$$

The meaning of (3.1) is that when expanded in the form

$$G(V, L)=\sum_{n=0}^{\infty}\left(\sum_{l} a_{nl}L^l\right)V^n \tag{3.2}$$

a_{nl} is equal to the number of states with angular momentum l in the representation $[n]$.

Then it is clear that a GF for the number of independent n-body interaction terms is given by

$$G_{\mathrm{U}(6)}(U)=\sum_{n=0}^{\infty}\left(\sum_{l} a_{nl}^{2}\right)U^{n} \tag{3.3}$$

whereas a GF for the number of independent Hermitian n-body terms is

$$H_{\mathrm{U}(6)}(U)=\sum_{n=0}^{\infty}\left(\sum_{l}\frac{a_{nl}(a_{nl}+1)}{2}\right)U^{n}. \tag{3.4}$$

We shall not explain the technical details of how (3.3) and (3.4) are explicitly derived from (3.1) but only give the final results:

$$G_{\mathrm{U}(6)}(U)=\frac{1+3U^{3}+6U^{3}+9U^{4}+6U^{5}+12U^{6}+6U^{7}+9U^{8}+6U^{9}+3U^{10}+U^{12}}{(1-U)^{2}(1-U^{2})^{3}(1-U^{3})^{2}(1-U^{4})} \tag{3.5}$$

$$H_{\mathrm{U}(6)}(U)=\frac{1+U^{2}+3U^{3}+4U^{4}+3U^{5}+6U^{6}+3U^{7}+5U^{8}+3U^{9}+2U^{10}}{(1-U)^{2}(1-U^{2})^{3}(1-U^{3})^{2}(1-U^{4})}. \tag{3.6}$$

The GF (3.5) and (3.6) provide a general and direct answer to the number of independent (Hermitian) n-body interaction terms in the interacting boson model, for all n. The expansions of (3.5) and (3.6) start as follows:

$$G_{\mathrm{U}(6)}(U)=1+2U+9U^{2}+24U^{3}+64U^{4}+140U^{5}+\ldots \tag{3.7}$$

$$H_{\mathrm{U}(6)}(U)=1+2U+7U^{2}+17U^{3}+41U^{4}+85U^{5}+\ldots \tag{3.8}$$

As a verification one sees that the numbers 9 and 7 for two-body terms (respectively 24 and 17 for three-body terms) are the same as found previously in this section.

Although the GF (3.6) generates a formula for the number of independent Hermitian n-body terms (n is arbitrary), it does not tell us exactly *which* n-body terms are actually independent. For one- and two-body interaction terms in the IBA Hamiltonian, this problem has been solved by Iachello (1980). The two independent one-body (or 'linear') terms can be chosen as $\hat{n}_s$ and $\hat{n}_d$, or equivalently as

$$\{\hat{N},\hat{n}_d\}. \tag{3.9}$$

A set of seven independent quadratic operators (i.e. two-body interaction terms) is given by

$$\{\hat{N}^{2},\hat{N}\hat{n}_d,\hat{n}_d^{2},C_{2,\mathrm{SO}(5)},C_{2,\mathrm{SO}(3)},C_{2,\mathrm{SU}(3)},C_{2,\mathrm{SO}(6)}\} \tag{3.10}$$

where $C_{k,L}$ is the kth order Casimir operator of L. Note that for symmetric representations $C_{1,\mathrm{U}(5)}=\hat{n}_d$ and $C_{2,\mathrm{U}(5)}=\hat{n}_d^{2}+4\hat{n}_d$. In a phenomenological analysis $\hat{N}$ has the constant eigenvalue N. Hence we deduce the well known fact that the most general IBA Hamiltonian up to two-body terms can be written completely in terms of the Hermitian SO(3) scalar operators contained in one of the subgroups SU(3), U(5) or SO(6). It is the aim of this paper to study higher-order terms and to investigate whether they can still be written as a linear combination of higher-order operators in the enveloping algebra of one of the three maximal subgroups. For this purpose, we shall first summarise the results for the structure of the degenerate enveloping algebras of the three subgroups SU(3), U(5) and SO(6).

4. Generating functions for SU(3), U(5) and SO(6)

In this section $G_L(U)$ is the notation for the GF for the number of independent SO(3) scalars in the (degenerate) enveloping algebra of the Lie algebra of L, and $H_L(U)$ is used for the corresponding GF for Hermitian SO(3) scalars.

The GF for the number of SO(3) scalars in the enveloping algebra of SU(3) has been determined by Judd *et al* (1974) and is given by equation (1) of I. Including Casimir operators, it becomes

$$G_{SU(3)}(U)=\frac{1+U^6}{(1-U^2)^2(1-U^3)^2(1-U^4)}. \tag{4.1}$$

Using a similar technique as in § 3, one finds the GF for the number of Hermitian SO(3) scalars:

$$H_{SU(3)}(U)=\frac{1}{(1-U^2)^2(1-U^3)^2(1-U^4)}. \tag{4.2}$$

The operators corresponding to the denominators in (4.2) are well known. The quadratic operators are $C_{2,SO(3)}$ and $C_{2,SU(3)}$; one third-order operator is the cubic Casimir operator $C_{3,SU(3)}$; the remaining cubic and quartic operators in the integrity basis are Ω and Λ, $X^{(3)}$ and $X^{(4)}$ in the notation of Moshinsky *et al* (1975), and these were successfully introduced in the IBA Hamiltonian by Vanden Berghe *et al* (1985).

The GF for SO(3) scalar operators in the degenerate enveloping algebra of U(5) follows from equation (10) of I:

$$G_{U(5)}(U)=\frac{1+3U^4+2U^5+3U^6+U^{10}}{(1-U)(1-U^2)^2(1-U^3)^2(1-U^4)}. \tag{4.3}$$

Using the same technique as in (3.1)-(3.4), we obtain

$$\begin{aligned}H_{U(5)}(U)&=\frac{1+U^4+U^5+U^6+U^{10}}{(1-U)(1-U^2)^2(1-U^3)^2(1-U^4)}\\&=1+U+3U^2+5U^3+\ldots.\end{aligned} \tag{4.4}$$

Clearly, the first-order operator is $\hat{n}_d$; the three second-order opeartors are $\hat{n}_d^2$, $C_{2,SO(5)}$ and $C_{2,SO(3)}$. The five third-order operators will be discussed in § 5.

For symmetric irreps of SO(6), only one Casimir operator is independent, namely $C_{2,SO(6)}$. Hence, equation (7) of I implies that the GF for the number of SO(3) scalars in the degenerate enveloping algebra of SO(6) is given by

$$G_{SO(6)}(U)=\frac{1+3U^4+U^5+3U^6+U^{10}}{(1-U^2)^3(1-U^3)^2(1-U^4)}. \tag{4.5}$$

The corresponding GF for the number of Hermitian SO(3) scalars is then

$$\begin{aligned}H_{SO(6)}(U)&=\frac{1+U^4+U^6+U^{10}}{(1-U^2)^3(1-U^3)^2(1-U^4)}\\&=1+3U^2+2U^3+\ldots.\end{aligned} \tag{4.6}$$

The three second-order operators are $C_{2,SO(6)}$, $C_{2,SO(5)}$ and $C_{2,SO(3)}$; the third-order operators will be discussed in § 5.

5. Cubic interaction terms respecting a dynamical symmetry

The GF obtained here predict the number of independent Hermitian SO(3) scalars of a certain degree in the (degenerate) enveloping algebra. Moreover, once the terms appearing in the numerator and the factors appearing in the denominator are identified, the GF also tells us which SO(3) scalars of a certain degree are independent. For example, knowing the denominators in (4.2), the GF implies that all higher-order terms are in fact products of the five operators $C_{2,\mathrm{SO}(3)}$, $C_{2,\mathrm{SU}(3)}$, $C_{3,\mathrm{SU}(3)}$, Ω and Λ. The identification of the operators appearing in the GF is, however, a rather difficult problem, even if we restrict ourselves to operators up to third order. Since computer programs are available (De Meyer *et al* 1987) which transform operators in the enveloping algebra of a Lie algebra into a certain chosen standard form, we have chosen this algebraic computing approach in order to compare operators and to find out which of them are independent. In this case we are interested in relations between operators acting on totally symmetric representations. This restriction can be built in the operators by realising them in terms of the boson operators $s^{(+)}$ and $d_\mu^{(+)}$. In other words, the degenerate enveloping algebra of U(6) is equal to the enveloping algebra of the Lie algebra spanned by $\{s, s^+, d_\mu, d_\mu^+, 1\}$ with non-vanishing commutation relations given by (2.2). It is the latter Lie algebra we use as input for the symbolic calculation programs.

Although the GF from §§ 3 and 4 are completely general for all higher-order terms, we shall from now on restrict ourselves to the analysis of cubic terms. For the SU(3) subalgebra, spanned by $Q^{(2)}$ and $T^{(1)}$, the following two cubic operators are independent:

$$[[Q^{(2)}\times Q^{(2)}]^{(2)}\times Q^{(2)}]^{(0)} \qquad [[T^{(1)}\times T^{(1)}]^{(2)}\times Q^{(2)}]^{(0)}. \tag{5.1}$$

The first operator can be replaced by the cubic SU(3) invariant $C_{3,\mathrm{SU}(3)}$. The second operator is equivalent to Ω (Moshinsky *et al* 1975); large parts of its eigenvalue spectrum have been obtained in algebraic closed form (De Meyer *et al* 1985, Vanden Berghe *et al* 1985).

The U(5) subalgebra is spanned by $\{\hat{n}_d, T^{(j)}; j=1,2,3,4\}$ and $\{T^{(1)}, T^{(3)}\}$ spans the SO(5) subalgebra. There are many ways to construct cubic SO(3) scalar operators by means of coupling three $T^{(j)}$ tensors to a tensor of rank zero. However, when acting on symmetric representations of U(5), only five cubic operators are independent, as shown by (4.4). By means of symbolic calculations it can be shown that the set of operators

$$\begin{array}{lll} \hat{n}_d^3 & [T^{(1)}\times T^{(1)}]^{(0)}\hat{n}_d & [T^{(2)}\times T^{(2)}]^{(0)}\hat{n}_d \\ \Lambda_1=[[T^{(1)}\times T^{(1)}]^{(2)}\times T^{(2)}]^{(0)} & & \Lambda_2=[[T^{(2)}\times T^{(2)}]^{(2)}\times T^{(2)}]^{(0)} \end{array} \tag{5.2}$$

constitutes a basis of Hermitian cubic SO(3) scalars. Note that the first operator is equivalent to $C_{3,\mathrm{U}(5)}$, and that the second and third operators can be replaced by $C_{2,\mathrm{SO}(3)}\hat{n}_d$ and $C_{2,\mathrm{SO}(5)}\hat{n}_d$. The last two operators Λ_1 and Λ_2 cannot be reformulated in terms of U(5), SO(5) or SO(3) invariants, nor as products of such invariants; just like Ω for SU(3) they are Hermitian SO(3) scalar operators in the enveloping algebra of U(5).

The SO(6) subalgebra, generated by $\{T^{(1)}, T^{(3)}, P^{(2)}\}$, also contains SO(5). In general, four Hermitian cubic SO(3) scalars can be constructed, namely

$$\begin{aligned}\Gamma_1&=[[P^{(2)}\times P^{(2)}]^{(2)}\times P^{(2)}]^{(0)}\\ \Gamma_2&=[[T^{(1)}\times T^{(1)}]^{(2)}\times P^{(2)}]^{(0)}+\mathrm{HC}\\ \Gamma_3&=[[T^{(3)}\times T^{(3)}]^{(2)}\times P^{(2)}]^{(0)}+\mathrm{HC}\\ \Gamma_4&=[[T^{(3)}\times T^{(1)}]^{(2)}\times P^{(2)}]^{(0)}+\mathrm{HC}\end{aligned} \tag{5.3}$$

where HC stands for the Hermitian conjugate. When acting on arbitrary SO(6) representations, all four operators (5.3) are independent. However, (4.6) shows that only two operators are independent for symmetric representations $(\sigma, 0, 0)$ of SO(6). Symbolic calculations show that the operators Γ_2, Γ_3 and Γ_4 become proportional when acting upon symmetric irreps $(\sigma, 0, 0)$. Hence, the two independent cubic interaction terms are Γ_1, on the one hand, and any linear combination of Γ_2, Γ_3 and Γ_4, on the other hand. For simplicity we can take Γ_2; this operator has been studied in more detail and parts of its spectrum have been obtained already (Vanthournout *et al* 1987). It should also be mentioned that the cubic part of the SO(6) invariant $C_{3,\mathrm{SO}(6)}$ is in general a linear combination of the scalars Γ_2, Γ_3 and Γ_4 alone. Moreover, when acting upon symmetric irreps this combination reduces to zero which proves that $C_{3,\mathrm{SO}(6)}$ in that case becomes of lower degree. This is in agreement with the fact that its eigenvalues are only quadratic in the representation label. As a consequence, none of the two independent cubic SO(3) scalars, say Γ_1 and Γ_2, can be reformulated in terms of invariants.

6. Cubic U(6) interaction terms and phenomenological parameters

From the GF (3.6) or (3.8) it follows that there are seventeen independent Hermitian SO(3) scalars in the degenerate U(6) enveloping algebra. By analysing the three subalgebra chains we have already found nine cubic scalars which respect a certain dynamical symmetry, i.e. which belong to the enveloping algebra of one of the maximal subalgebras. However, as elements of the enveloping algebra of U(6), only eight of the nine operators (5.1), (5.2), Γ_1 and Γ_2 are independent. Indeed, it follows from (2.3) that

$$Q^{(2)}=P^{(2)}-\tfrac{1}{2}\sqrt{7}T^{(2)}. \tag{6.1}$$

Hence $\Omega=[[T^{(1)}\times T^{(1)}]^{(2)}\times Q^{(2)}]^{(0)}$ in the SU(3) enveloping algebra is a linear combination of $\Gamma_2=[[T^{(1)}\times T^{(1)}]^{(2)}\times P^{(2)}]^{(0)}$ and $\Lambda_1=[[T^{(1)}\times T^{(1)}]^{(2)}\times T^{(2)}]^{(0)}$ occurring in the enveloping algebras of SO(6) and U(5) respectively. Hence, we have to drop one of the three operators Ω, Γ_2 or Λ_1, for instance the last one.

Seven more independent cubic scalars follow by multiplying the seven independent quadratic scalars by $\hat{N}=C_{1,\mathrm{U}(6)}$. Finally, the two remaining independent scalars may be chosen as products of invariants belonging to different dynamical subalgebra chains, for example $\hat{n}_d C_{2,\mathrm{SU}(3)}$ and $\hat{n}_d C_{2,\mathrm{SO}(6)}$. Of course, many other choices are possible, but it always turns out that at least two mixed-symmetry operators are necessary in order to find a set of seventeen independent cubic scalars.

The most general Hamiltonian of cubic interaction terms is then

$$\begin{aligned}H^{(3)}=\hat{N}(&c_1\hat{N}^2+c_2\hat{N}\hat{n}_d+c_3\hat{n}_d^2+c_4C_{2,\mathrm{SO}(5)}+c_5C_{2,\mathrm{SO}(3)}+c_6C_{2,\mathrm{SU}(3)}+c_7C_{2,\mathrm{SO}(6)})\\ &+c_8C_{3,\mathrm{SU}(3)}+c_9\hat{n}_d^3+c_{10}\hat{n}_dC_{2,\mathrm{SO}(3)}+c_{11}\hat{n}_dC_{2,\mathrm{SO}(5)}+c_{12}\hat{n}_dC_{2,\mathrm{SU}(3)}\\ &+c_{13}\hat{n}_dC_{2,\mathrm{SO}(6)}+c_{14}\Omega+c_{15}\Lambda_2+c_{16}\Gamma_1+c_{17}\Gamma_2.\end{aligned} \tag{6.2}$$

In a phenomenological analysis of IBA, $\hat{N}$ has the constant eigenvalue N and this reduces the number of terms in the Hamiltonian considerably. For instance,

$$H^{(1)} = a_1\hat{N} + a_2\hat{n}_d \tag{6.3}$$

contains two parameters, but only one phenomenological parameter (a_2). Similarly,

$$H^{(2)} = b_1\hat{N}^2 + b_2\hat{N}\hat{n}_d + b_3\hat{n}_d^2 + b_4C_{2,\mathrm{SO}(5)} + b_5C_{2,\mathrm{SO}(3)} + b_6C_{2,\mathrm{SU}(3)} + b_7C_{2,\mathrm{SO}(6)} \tag{6.4}$$

contains seven parameters, but only five phenomenological parameters ($b_3, b_4, \ldots, b_7$) since $\hat{N}^2 = N^2$ and $\hat{N}\hat{n}_d$ reduces to a lower-order operator. It follows from (6.2) that $H^{(3)}$ contains only ten phenomenological parameters ($c_8, c_9, \ldots, c_{17}$). Eight of the ten remaining cubic terms respect one of the dynamical symmetries, whereas two terms clearly mix dynamical symmetries. Also, note that four of the ten terms cannot be written as group or subgroup invariants, nor as a product of such invariants. In this respect, the three-body interaction is completely different compared to the standard linear and two-body IBA Hamiltonian.

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References

Arima A and Iachello F 1976 *Ann. Phys., NY* **99** 253–317
—— 1978 *Ann. Phys., NY* **111** 201–38
—— 1979 *Ann. Phys., NY* **123** 468–92
De Meyer H, Vanden Berghe G and De Wilde P 1987 *Comp. Phys. Commun.* to be published
De Meyer H, Vanden Berghe G and Van der Jeugt J 1985 *J. Math. Phys.* **26** 3109–11
Gaskell R, Peccia A and Sharp R T 1978 *J. Math. Phys.* **19** 727–33
Giroux Y, Couture M and Sharp R T 1984 *J. Phys. A: Math. Gen.* **17** 715–25
Iachello F 1980 *Nuclear Structure* ed K Abrahams, K Allaart and A E L Dieperink (New York: Plenum) pp 53–89
Judd B R, Miller W Jr, Patera J and Winternitz P 1974 *J. Math. Phys.* **15** 1787–99
Moshinsky M, Patera J, Sharp R T and Winternitz P 1975 *Ann. Phys., NY* **95** 139–69
Vanden Berghe G, De Meyer H and Van Isacker P 1985 *Phys. Rev.* C **32** 1049–56
Van der Jeugt J 1986 *J. Phys. A: Math. Gen.* **19** L463–6
Vanthournout J, Van der Jeugt J, De Meyer H and Vanden Berghe G 1987 *J. Math. Phys.* submitted